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An exact formulation of inventory costs and optimal lot size in face of sudden obsolescence

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Recently, approximate formulas for determining the relevant total costs and the optimal lot size in face of sudden obsolescence were proposed. In this paper, we provide an exact formulation of the relevant costs which, when minimized, give the true optimal lot size. The estimation errors of the approximate model are significant in some situations, and the use of our exact model is recommended.

sudden obsolescence; inventory models

Introduction

With rapidly evolving technology, abrupt changes in global political situations, and almost instantaneous dissemination of information in the market place, today a number of products are at risk of becoming obsolete overnight. Swiss watches, anti-USSR movies, breast implants, and Milli Vanilli records are examples of the phenomenon of sudden obsolescence. Although obsolescence can also set in gradually, this paper focuses on the implications of sudden obsolescence for inventory-related decisions.

The traditional approach to inventory related decisions in face of obsolescence is to add a risk component to the product's cost of holding [2,4,5,6]. While costs of capital, storage, insurance, deterioration, pilferage, and taxes, which constitute the other components of inventory holding costs, do indeed accrue over time, and can be logically applied to the average amount of inventory held, obsolescence costs are different. They accrue at random points in time, and apply to the entire inventory on hand at the time obsolescence sets in. Therefore, as Hadley and Whitin [1] have argued, it is inappropriate to include obsolescence costs as a component of holding costs, and one needs a mathematical model that explicitly recognizes the obsolescence phenomenon for what it is.

Recently, retaining standard assumptions of the Economic Order Quantity (EOQ) model, but allowing for sudden obsolescence, Masters [3] developed an approximate formula for determining the total costs of inventory ordering, carrying, and obsolescence resulting from a time-invariant lot size. Using this

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approximate formula, Masters [3] claimed that the traditional approach is appropriate for products subject to sudden obsolescence.

In this paper, we develop an exact formulation of the relevant total costs resulting from a given lot size policy. This formulation enables us to gauge the errors of approximation in Masters' [3] model under a variety of circumstances. We find that Masters' approximation consistently (a) underestimates true costs, and (b) recommends a larger than optimal lot size.

In situations involving a large number of inventory cycles during a product's lifetime, the errors in Masters' model are practically insignificant. Even in situations involving a small number of lifetime inventory cycles, the use of Masters' optimal lot size may not be very costly, but only because the EOQ model is so insensitive to small errors in the lot size. However, the use of Masters' model for estimating lifetime costs in such situations would lead to serious underestimation of the true costs, and the use of our exact model is recommended. Finally, we believe that with the computer facilities available at most firms today, perhaps there is no reason why one should depend on an approximate model when an exact model is available.

A recapitulation of masters' model

Masters [3] defines 'sudden obsolescence' as a situation when a product's lifetime is described by a negative exponential distribution. As Masters explains,

"Although many distributions are possible, the exponential is appropriate for the sudden obsolescence phenomenon since it models a constant obsolescence rate; that is, the case where the age of the item does not influence the probability of obsolescence during any subsequent interval. This is not to suggest that all item lifetimes necessarily follow the exponential. Rather, the phenomenon of sudden obsolescence is defined to be the situation which arises when an item has a constant obsolescence rate." [3, p. 118]

Masters also observes that in situations of sudden obsolescence, a time-invariant lot size is optimal. This is because when the obsolescence rate is constant, the process has no memory. Consequently, at the beginning of any inventory cycle (provided that the product has not become obsolete already), the expected future costs resulting from a lot size of Q units is exactly the same as the expected future costs of that lot size at the beginning of any other cycle. Thus the optimal lot size does not depend on the age of the product.

Retaining standard assumptions of the EOQ model, Masters [3] considers a product whose demand vanishes abruptly and completely at a random point in time, t, where t is negative exponentially distributed with a mean value of L years. In the general form of his model, Masters assumes that any stock remaining on hand at time t is obsolete, but has some salvage value. The cost of obsolescence per unit is the difference between the product's unit cost and the salvage value. Masters' notation can be summarized as below:

t = Time at which the product becomes obsolete.
L = Expected life of the product (years).
Q = Time-invariant lot size.
D = Demand in units per year until obsolescence.
C_o = Ordering costs per order.
H = Annual holding costs (other than obsolescence) per dollar of inventory held.
C_p = Unit cost of the product.
C_s = Cost of obsolescence per unit of inventory on hand at time t.
$\mathrm{INT}[x]$ = The largest integer $\leq x$.
$\mathrm{E}[TC(Q)]$ = Expected total lifetime costs of ordering, holding, and obsolescence resulting from a lot size of Q.

In estimating the expected ordering costs, Masters argues that in a given lifetime of t years, there will be $1 + \mathrm{INT}[(tD/Q)]$ total orders; hence the expected number of orders over a product lifetime equals

E{1 + INT[(tD/Q)]}. In his Table 1 (not reproduced here), Masters compares various values of (LD/Q) with the corresponding values of E{1 + INT[(tD/Q)]}. Although for $(LD/Q)=1$, the value of E{1 + INT[(tD/Q)]} is 1.582, for a percent difference of 36.788, Masters ignores it. Instead, arguing that the percent differences are small, and the absolute differences are nearly constant across a broad range of values of (LD/Q), ranging from 10 to 200, Masters suggests that E{1 + INT[(tD/Q)]} can be well approximated as (LD/Q). In estimating expected lifetime holding and obsolescence costs, Masters makes another assumption, namely that the expected inventory on hand at a random point in time will be $Q/2$. With these two assumptions, Masters' formula for expected lifetime costs is

$$E[TC(Q)] = LDC_o/Q + LHC_pQ/2 + C_sQ/2 \tag{1}$$

and his formula for the optimal lot size, Q^*, is

$$Q^* = \{[2DC_o]/[HC_p + (C_s/L)]\}^{1/2}. \tag{2}$$

Clearly then, in situations where Masters' assumptions are reasonably valid, the traditional approach of incorporating obsolescence costs as a component of holding costs is appropriate. All one has to do is to calculate the per-unit-per-year holding costs by the formula $[(HC_p) + (C_s/L)]$, and the standard EOQ model would yield the optimal Q^*, the corresponding lifetime costs being given by (1). From an implementation point of view, permitting an adaptation of the standard EOQ model is the major appeal of Masters' model.

However, in order to determine when Masters' assumptions are reasonable and when they are not, we must first have an exact formula for the lifetime costs resulting from a given lot size policy. The following section develops this exact formula.

An exact formulation of expected lifetime costs

In addition to the foregoing notation, we now define

C_c = Costs of ordering, holding, and obsolescence during an order cycle of Q units, as expected at the beginning of that order cycle

When an inventory cycle with an order quantity of Q units begins, the ordering cost of C_o is incurred with certainty. However, the costs of holding and obsolescence depend upon whether, and when, the product becomes obsolete during the inventory cycle.

If the product becomes obsolete at time t $(t \le Q/D)$, then at that time the inventory on hand is $[Q-(tD)]$. Therefore, the obsolescence cost during an inventory cycle, as expected at the beginning of the cycle, is given by

$$\int_0^{Q/D} [Q-(tD)](C_s)[(1/L)\ e^{-(t/L)}]\ dt. \tag{3}$$

Furthermore, if $t \le Q/D$, since the inventory at the beginning of the cycle is Q units, and the inventory at time t is $[Q-(tD)]$, the average inventory during time t is $(\frac{1}{2})\{(Q)+[Q-(tD)]\}$. Consequently, the corresponding holding costs, as expected at the beginning of the cycle, are given by

$$\int_0^{Q/D} [Q-(tD/2)](C_pHt)[(1/L)\ e^{-(t/L)}]\ dt. \tag{4}$$

On the other hand, if the product does not become obsolete during an order cycle, [i.e., if $t \ge (Q/D)$], then, as in the standard EOQ model, the beginning inventory is Q units, the ending inventory is zero, and the average inventory of $(Q/2)$ units is carried for a period of (Q/D) years during the inventory cycle. Consequently, the corresponding holding costs during the inventory cycle, as expected at the beginning of an inventory cycle, are given by

$$\int_{Q/D}^{\infty} [(Q/2)(Q/D)(C_pH)][(1/L)\ e^{-(t/L)}]\ dt. \tag{5}$$

Thus, C_c, the total inventory-related costs during an order cycle, are given by

$$C_c = C_o + \int_0^{Q/D} [Q - (tD)](C_s)[(1/L)\ e^{-(t/L)}]\ dt$$
$$+ \int_0^{Q/D} [Q - (tD/2)](C_p Ht)[(1/L)\ e^{-(t/L)}]\ dt$$
$$+ \int_{Q/D}^{\infty} [(Q/2)(Q/D)(C_p H)][(1/L)\ e^{(t/L)}]\ dt. \tag{6}$$

After appropriate integration and algebraic simplification (see Appendix A), we obtain

$$C_c = C_o + [HC_p L + C_s][Q - DL(1 - e^{-[Q/(DL)]})]. \tag{7}$$

Let us further define

P_s = Probability, at the beginning of an order cycle of Q units, that the product does not become obsolete during the order cycle.

C_L = Lifetime costs of ordering, holding, and obsolescence, as expected at the beginning of any order cycle of Q units.

Observe that our definition of expected lifetime costs is different than Masters' definition. Masters assumes that expected lifetime costs at any random point in time are the same. However, it must be recognized that expected lifetime costs are different at different points in an inventory cycle. For example, the expected future costs at the midpoint of an inventory cycle, assuming that obsolescence has not occurred until then, would be quite different than the expected future costs at the beginning of an order cycle.

On the other hand, because of the assumed negative exponential distribution of the product lifetime, the process has no memory. Consequently, at the beginning of any inventory order cycle, the expected future pattern of obsolescence, and hence the expected inventory-related costs influenced by that pattern, will be identical to those at the beginning of any other cycle. In other words, for a given Q, C_L will be a constant regardless of the age of the product. Thus, the relationship between C_c and C_L can be expressed as

$$C_L = C_c + C_L P_s. \tag{8}$$

That is,

$$C_L = C_c/[1 - P_s]. \tag{9}$$

Note also that

$$P_s = \int_{Q/D}^{\infty} [(1/L)\ e^{-(t/L)}]\ dt = e^{-[Q/(DL)]}. \tag{10}$$

Consequently, C_L can be written as

$$C_L = C_o/(1 - e^{[Q/(DL)]}) + [HC_p L + C_s]\{[Q/(1 - e^{-[Q/(DL)]})] - DL\} \tag{11}$$

Equation (11) gives the exact lifetime inventory costs of a product subject to sudden obsolescence, as expected at the beginning of any order cycle with Q as the lot size. A comparison of (11) with (1) clearly shows (see Appendix B) that Masters' approximation underestimates the true lifetime costs resulting from a given Q. Numerical examples in the next section indicate the magnitudes of this underestimation.

Now, to obtain the optimal lot size, Q^*, we should minimize (11) with respect to Q. There is no closed-form solution to this minimization problem. However, for specific numerical values of the parameters, the software package GINO solves the problem within seconds on a microcomputer. Thus, given the computational facilities at most firms today, perhaps there is no need for an approximate model.

On the other hand, managers are often reluctant to implement a model that is difficult to understand. Therefore, an intuitively appealing approximation, particularly if the associated errors are small, may be preferred. Masters' model certainly qualifies as an intuitively appealing one. However, numerical examples in the next section suggest that in certain situations, the errors in Masters' [3] estimates may be significant, and the use of the exact model may be necessary.

Numerical examples

In Table 1, we present three scenarios that span a broad range of values of the parameters in our model with progressively smaller expected numbers of inventory cycles during a product's lifetime. Table 1 presents the assumptions as well as the results of each scenario in a self-explanatory manner. As can be seen, Masters' model consistently recommends larger than true optimal lot size, and underestimates the true costs of a given lot size.

In the third scenario, expected lifetime is only three months, ordering costs are substantial, and obsolescence costs are 90% of the unit cost. Consequently, the expected number of order cycles during

Table 1
Comparison of the exact model with Masters' model for three scenarios

Assumptions and results	Scenario 1	Scenario 2	Scenario 3
ASSUMPTIONS:			
D = Demand (units per year)	10,000	10,000	10,000
L = Expected lifetime (years)	10	2	0.25
C_0 = Ordering cost ($/order)	10	100	1,000
C_p = Unit cost ($/unit)	5	5	5
C_s = Obsolescence cost ($/unit)	2.5	3.5	4.5
H = Holding cost ($/dollar of inventory/yr)	0.2	0.2	0.2
RESULTS:			
Q^* = Optimal lot size (units/order) as given by the exact model	399.73	846.79	960.35
Q_m^* = Optimal lot size (units/order) as given by Masters' model	400.00	852.80	1,025.98
Error in Q_m^*			
absolute (units/order)	0.27	6.01	65.63
percent $[(Q_m^* - Q^*)/Q_m^*]$	0.07%	0.70%	6.40%
$CL(Q^*)$ = Lifetime costs at true Q^* as given by the exact model	5,006.67	4,757.32	5,561.65
$CL_m(Q^*)$ = Lifetime costs at true Q^* as given by Masters' model	5,000.00	4,690.53	4,884.05
Error in $CL_m(Q^*)$			
absolute ($)	−6.67	−66.79	−677.6
percent $[(CL_m(Q^*) - CL(Q^*))/CL_m(Q^*)]$	−0.13%	−1.42%	−13.87%
$CL(Q_m^*)$ = Lifetime costs at Masters' Q_m^* as given by the exact model	5,010.02	4,757.44	5,573.71
$CL_m(Q_m^*)$ = Lifetime costs at Masters' Q_m^* as given by Master's model	5,000.00	4,690.42	4,873.40
Error in $CL_m(Q_m^*)$			
absolute ($)	−10.02	−67.02	−700.31
percent $[(CL_m(Q_m^*) - CL(Q_m^*))/CL_m(Q_m^*)]$	−0.20%	−1.43%	−14.3%
LD/Q^* = Expected number of order cycles as given by the exact model	250.17	23.62	2.60

the product's lifetime is only 2.6. In this case, Masters' optimal lot size is 6.4% larger than the true optimal lot size, and his model underestimates the true costs by approximately 14%. Some readers may be skeptical of our assumptions in Scenario 3. However, we believe that these assumptions are not unrealistic for a bookseller chain (such as Encore Books) that is deciding how many copies of a new novel to stock, or for a music store chain (such as Sam Goody) that is deciding how many copies of a new record to stock. Indeed, we believe that Scenario 3 truly captures the phenomenon of sudden obsolescence, and shows that in such situations, cost estimates should be based only on the exact model.

Of course, in conformance with the well known fact that the EOQ model is insensitive to small errors in the lot size, our numerical examples show that the use of Masters' model for determining the lot size will not affect the true lifetime costs significantly. However, the use of Masters' lifetime cost formula would lead to a serious underestimation.

Appendix A. Derivation of equation (7) from equation (6)

First, let us define

$$\boldsymbol{k} = \mathrm{e}^{-[Q/(DL)]}.$$

We note the following finite integrals:

$$\int_0^{Q/D} \left[(1/L)\ \mathrm{e}^{-(t/L)}\right] \mathrm{d}t = 1 - \boldsymbol{k}, \tag{A.1}$$

$$\int_{Q/D}^{\infty} \left[(1/L)\ \mathrm{e}^{-(t/L)}\right] \mathrm{d}t = \boldsymbol{k}. \tag{A.2}$$

$$\int_0^{Q/D} \left[(1/L)\ \mathrm{e}^{-(t/L)}\right](t)\ \mathrm{d}t = L(1-\boldsymbol{k}) - (Q/D)\boldsymbol{k} \quad \text{and} \tag{A.3}$$

$$\int_0^{Q/D} \left[(1/L)\ \mathrm{e}^{-(t/L)}\right](t^2)\ \mathrm{d}t = 2L^2(1-\boldsymbol{k}) - [2L + (Q/D)](Q/D)\boldsymbol{k}. \tag{A.4}$$

Using these integrals in (6), we obtain

$$C_c = C_o + (QC_s)(1-\boldsymbol{k}) - (DC_s)[L(1-\boldsymbol{k}) - (Q/D)\boldsymbol{k}] + (QC_pH)[L(1-\boldsymbol{k}) - (Q/D)\boldsymbol{k}]$$
$$- (DC_pH/2)\{2L^2(1-\boldsymbol{k}) - [2L + (Q/D)](Q/D)\boldsymbol{k}\} + \left[\left(Q^2C_pH\right)/(2D)\right]\boldsymbol{k}. \tag{A.5}$$

That is,

$$C_c = C_o + QC_s - QC_s\boldsymbol{k} - (DLC_s)(1-\boldsymbol{k}) + QC_s\boldsymbol{k} + (QC_pHL)(1-\boldsymbol{k}) - \left(Q^2C_pH/D\right)\boldsymbol{k}$$
$$- \left(DC_pHL^2\right)(1-\boldsymbol{k}) + QC_pHL\boldsymbol{k} + \left[\left(Q^2C_pH\right)/(2D)\right]\boldsymbol{k} + \left[\left(Q^2C_pH\right)/(2D)\right]\boldsymbol{k}. \tag{A.6}$$

Eliminating mutually canceling terms,

$$C_c = C_o + QC_s - (DLC_s)(1-\boldsymbol{k}) + QC_pHL - \left(DC_pHL^2\right)(1-\boldsymbol{k}). \tag{A.7}$$

Thus,

$$C_c = C_o + \left[HC_pL + C_s\right]\left[Q - DL(1-\boldsymbol{k})\right]. \tag{A.8}$$

Substituting back $\boldsymbol{k} = \mathrm{e}^{-[Q/(DL)]}$ in (A.8), we obtain (7).

Appendix B. Proof that Masters' approximation underestimates the true life-time costs

Lifetime costs of holding and obsolescence

From a comparison of (1) and (11), it is clear that Masters' approximation underestimates the lifetime costs of holding and obsolescence, if

$$\left[HC_pL + C_s\right](Q/2) < \left[HC_pL + C_s\right]\left\{\left[Q/(1 - e^{-[Q/(DL)]})\right] - DL\right\}. \tag{B.1}$$

That is, if

$$(Q/2) < \left[Q/(1 - e^{-[Q/(DL)]})\right] - DL \tag{B.2}$$

or, if

$$(1/2)(Q/DL) < \left[(Q/DL)/(1 - e^{-[Q/(DL)]})\right] - 1. \tag{B.3}$$

To simplify, substitute y for (Q/DL). Observe that $y \geq 0$.
We want to show that

$$(1/2)y < \left[y/(1 - e^{-y})\right] - 1. \tag{B.4}$$

That is,

$$y(1 - e^{-y}) < 2y - 2(1 - e^{-y}) \quad \text{or,} \tag{B.5}$$

$$2(1 - e^{-y}) < y[2 - (1 - e^{-y})] \quad \text{or,} \tag{B.6}$$

$$2(1 - e^{-y}) < y[1 + e^{-y}]. \tag{B.7}$$

Multiplying both sides of (B.7) by e^y, we have

$$2(e^y - 1) < y[e^y + 1]. \tag{B.8}$$

Using MacLaurin's expansion for e^y in (B.8), we have

$$2\left[(1 + y + y^2/2! + y^3/3! + y^4/4! + y^5/5! + \cdots) - 1\right] < y\left[(1 + y + y^2/2! + y^3/3! + y^4/4! + y^5/5! + \cdots) + 1\right]. \tag{B.9}$$

Thus, we want to show that

$$2\left[y + y^2/2! + y^3/3! + y^4/4! + y^5/5! + \cdots\right] < y\left[2 + y + y^2/2! + y^3/3! + y^4/4! + y^5/5! + \cdots\right] \tag{B.10}$$

or,

$$2y^3/3! + 2y^4/4! + 2y^5/5! + \cdots < y^3/2! + y^4/3! + y^5/4! + \cdots. \tag{B.11}$$

Now, since each term on the left hand side is smaller than the corresponding term on the right hand side, clearly, (B.11) is true. Thus, we have shown that Masters' approximation underestimates the true lifetime costs of holding and obsolescence.

Lifetime costs of ordering

From a comparison of (1) and (11), it is clear that Masters' approximation underestimates the lifetime costs of ordering, if

$$C_o[DL/Q] < C_o/(1 - e^{-[Q/(DL)]}). \tag{B.12}$$

That is, if

$$DL < Q/(1 - e^{-[Q/(DL)]}) \tag{B.13}$$

or, if

$$0 < \left[Q/(1 - e^{-[Q/(DL)]}) \right] - DL. \tag{B.14}$$

But (B.14) will be true whenever (B.2) is true, and we have already shown that (B.2) is true. Thus, Masters' approximation underestimates the lifetime costs of ordering, just as it underestimates the lifetime costs of holding and obsolescence.

References

[1] G. Hadley and T.M. Whitin. *Analysis of Inventory Systems*, Prentice Hall, Englewood Cliffs, NJ, 1963.
[2] D.M. Lambert and B.J. LaLonde, "Inventory carrying costs", *Management Accounting* **58**/2, 31–35 (1976).
[3] J.M. Masters, "A note on the effect of sudden obsolescence on the optimal lot size", *Decision Sciences* **22**/5, 1180–1186 (1991).
[4] E. Naddor, *Inventory Systems*, John Wiley and Sons, New York, 1966.
[5] E.A. Silver and R. Peterson, *Decision Systems for Inventory Management and Production Planning*, John Wiley and Sons, New York, 1985.
[6] R.J. Tersine, *Principles of Inventory and Materials Management*, Elsevier Science Publishing Co., New York, 1988.