

CHAPTER 6: HEISENBERG UNCERTAINTY PRINCIPLE

Cannot determine the exact position and momentum of a particle at the same time

\[ \Delta x \cdot \Delta p \geq \frac{\hbar}{2} \]

\( \Delta x \) = uncertainty in determining position
\( \Delta p \) = uncertainty in determining momentum

Can be restated as \( \sigma_x \cdot \sigma_p \geq \frac{\hbar}{2} \) where \( \sigma_x \) & \( \sigma_p \) are standard deviations

Similarly, cannot exactly measure energy without taking infinite amount of time for measurement

In spectroscopy, the shorter the time that you measure an energy, the more the uncertainty

\[ \Delta E \cdot \Delta t \geq \frac{\hbar}{2} \]

\( \Delta E \) = uncertainty in determining energy
\( \Delta t \) = uncertainty in determining time

For momentum – can also swap in other classical equations

For \( p = (\text{mass})(\text{velocity}) \) get \( \Delta p = m \Delta v \)

For \( KE = \frac{p^2}{2m} \) get \( \Delta p = (2m \cdot \Delta KE)^{1/2} \)

Heisenberg Uncertainty Principle predated the Schrödinger equation. Was part of ongoing process in the early 1900’s of sorting out duality of matter being particles or waves

Semi-classical Interpretation

To accurately measure particle position, need to take a snapshot in time, freezing the particle in place.

However, freezing the particle in place to measure its position means that there is no observable motion, hence no way to measure momentum.

Alternatively, to measure momentum, you must allow the particle to move some.

And the longer the particle is allowed to move, then the more accurate the measurement of \( p \).

But, the longer the particle is allowed to move, then the less accurate is your ability to measure \( x \).

Heisenberg’s Measurement Interpretation

The very act of measuring a system perturbs the system

Quantum mechanically, an unobserved electron’s location and other observable properties are described by the probability information in \( \psi \).

Once the particle is observed, however, then full wavefunction is no longer fully utilized and hence the observable information of the wavefunction begins to be lost.
Superposition Interpretation – Wave Packets

A single eigenfunction of Schrödinger Eqn. (for example $A \cdot \sin(kx)$ for particle-in-a-box) covers span of all allowed space, with $x$ being found anywhere along function. To specify a single position, need to build up series of eigenfunctions to describe a new function that is sharply specified in location, but is no longer an eigenfunction. Each of those contributing basis functions contribute their own momentum to the average. Localized functions constructed from multiple eigenfunctions are called “wave packets”.

![Desired Function to Exactly Locate Particle](image1)

![Eigenfunctions to Use as Basis Set](image2)

![Linear Combination of Basis Functions](image3)

<table>
<thead>
<tr>
<th>Quantum number, $n$</th>
<th># of functions</th>
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<tr>
<td>1</td>
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Commuting vs. Complementary Pairs of Operators

Commuting operators can swap their order: $[\hat{O}_1, \hat{O}_2] = 0$

Complementary (or non-commuting) operators are those whose order do not commute

Application to Heisenberg Uncertainty Principle

If operators do not commute, then their observables are limited by uncertainty principle. (i.e. cannot know both properties at same time)

E.g., $\hat{x}$ & $\hat{p}$ do not commute, $[\hat{x}\hat{p}, \hat{p}\hat{x}] = h\psi / \hbar \neq 0$ hence uncertainty for $x$ and $p$.

Similarly, $\hat{H}$ and $\hat{x}$ do not commute, so cannot know both $E$ and $x$ at same time

Example:

For momentum and position: $\hat{O}_p = \frac{\hbar}{i} \frac{d}{dx}$ and $\hat{O}_x = \text{multiply by } x$

If $\psi(x) = \sin(kx)$

$$\hat{O}_p \cdot \left[ \hat{O}_x \psi(x) \right] - \hat{O}_x \cdot \left[ \hat{O}_p \psi(x) \right] = 0$$

$$\frac{\hbar}{i} x \left[ \frac{\partial}{\partial x} \psi \right] - \psi = 0$$

$$\frac{\hbar}{i} \left[ x \frac{d\psi}{dx} + \psi \frac{dx}{dx} \right] - \psi = 0$$

Get different answer if change the order of the operators, therefore position and momentum are non-commuting, and cannot know both at the same time.
Zero-Point Energy

“Zero-Point Energy” = energy of the lowest possible quantum state
This zero point energy cannot be zero. Must have some finite non-zero energy value

A corollary to Heisenberg Uncertainty Principle is that all quantum states must have:
- non-zero energy
- and non-zero probability distribution

For Particle in a Box, this means that no \( n=0 \) state is possible
If there was an \( n=0 \) state, then that means \( \psi(x) = 0 \) everywhere in space,
and hence there is 0 probability of finding particle anywhere in space.
It also would require \( E_n = 0 \), which would mean no kinetic energy, or the particle stopped
(non-moving) in space, which would mean that we could know its position exactly.

\[ E_n = \frac{\hbar^2 n^2}{8 m L^2} \]

Motion at Temperature = Absolute Zero
Another implication of zero-point energy is that since this is true of all quantum systems, it must be that atoms in a crystal lattice at zero Kelvin (absolute zero) must have some level of motion associated with them.
I.e. atoms never come completely to rest and form a perfect crystal even at \( T = 0 \) K.

Particle Physics
The “Standard Model” of particle physics requires virtual particles to appear and disappear in space and time. These are regular particles, but are transient in time.
According to Einstein’s Special Relativity, \( E = mc^2 \)
If Heisenberg Uncertainty is true, then \( \Delta E \cdot \Delta t \geq \frac{1}{2} \hbar \)
Combining Relativity and Heisenberg, you can CREATE NEW MASS, so long as it doesn’t stay around too long (\( \Delta t \) is small)